7.1 Laplace Transform

the definition of Laplace transform of $f(t)$ is

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\underbrace{e^{-st}}_{\text{kernel of transform}}\,dt}$$

Laplace transform is one type of integral transform

(another commonly used transform is the Fourier transform)

let's look at the transforms of some simple functions

$f(t) = 1$

s is "constant"

$$\mathcal{L}\{1\} = F(s) = \int_0^{\infty} 1 \cdot e^{-st}\,dt = \lim_{a\to\infty}\int_0^{a} 1\cdot e^{-st}\,dt$$

← variable

$$= \lim_{a\to\infty} -\frac{1}{s}e^{-st}\Big|_{t=0}^{t=a}$$

$$= \lim_{a\to\infty}\left(-\frac{1}{s}\underbrace{e^{-sa}}+\frac{1}{s}\right)$$

must go to zero for integral to converge $\rightarrow$ $s>0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}\ ,\ s>0}$$

$$\mathcal{L}\{t\} = F(s) = \int_0^\infty t\cdot e^{-st}\,dt = \lim_{a\to\infty}\int_0^a t\,e^{-st}\,dt$$

by parts $u=t$ $\quad dv=e^{-st}dt$

$du=dt$ $\quad v=-\frac{1}{s}e^{-st}$

$uv-\int v\,du$

$$= \lim_{a\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_0^a + \int_0^a \frac{1}{s}e^{-st}\,dt\right)$$

$$= \lim_{a\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_0^a - \frac{1}{s^2}e^{-st}\Big|_0^a\right)$$

$$= \lim_{a\to\infty}\left(-\frac{a}{s}\underbrace{e^{-sa}} - \frac{1}{s^2}\underbrace{e^{-sa}} + \frac{1}{s^2}\right)$$

must go to zero
for integral to converge, so $s>0$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0}$$

using the same process, we find $\mathcal{L}\{t^2\} = \frac{2}{s^3}$

$$\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a t^3 e^{-st}\,dt \qquad \begin{array}{ll} u = t^3 & dv = e^{-st}dt \\ du = 3t^2 dt & v = -\frac{1}{s}e^{-st} \end{array}$$

$$= \lim_{a\to\infty} \Bigg(\underbrace{-\frac{t^3}{s}e^{-st}\Big|_0^a}_{\substack{\to 0 \text{ if} \\ s>0}} + \int_0^a \frac{3t^2}{s} e^{-st}\,dt \Bigg)$$

$$= \frac{3}{s} \underbrace{\int_0^\infty t^2 e^{-st}\,dt}_{\mathcal{L}\{t^2\}} = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3\cdot 2}{s^4}$$

repeat and eventually we get

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0}$$

$$\mathcal{L}\{e^t\} = \lim_{a\to\infty} \int_0^a e^t e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a e^{(1-s)t}\,dt$$

$$= \lim_{a\to\infty} \frac{1}{1-s} e^{(1-s)t} \Big|_0^a$$

$$= \lim_{a\to\infty} \frac{1}{1-s} \underbrace{e^{(1-s)a}}_{\text{must go to } 0} - \frac{1}{1-s} = \frac{1}{s-1}$$

$$1-s < 0$$

$$s > 1$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \iff s > a$$

Laplace transform is linear because integration is linear

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty [f(t)+g(t)]\,e^{-st}\,dt$$

$$= \int_0^\infty f(t)\,e^{-st}\,dt + \int_0^\infty g(t)\,e^{-st}\,dt$$

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{c\,f(t)\} = c\cdot\mathcal{L}\{f(t)\}$$

for example, $\mathcal{L}\{3+5e^{2t}-10t^2\}$

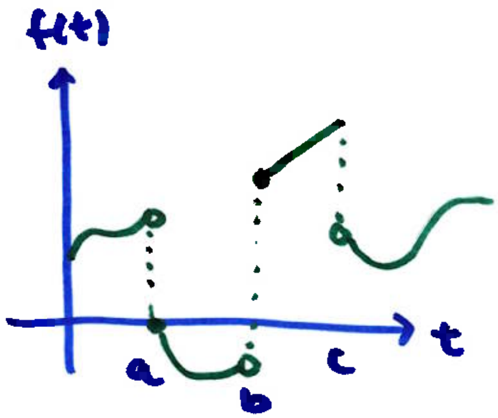
$$= \mathcal{L}\{3\} + \mathcal{L}\{5e^{2t}\} - \mathcal{L}\{10t^2\}$$

$$= 3\mathcal{L}\{1\} + 5\mathcal{L}\{e^{2t}\} - 10\mathcal{L}\{t^2\}$$

$$= 3\cdot\frac{1}{s} + 5\cdot\frac{1}{s-2} - 10\cdot\frac{2}{s^3}, \quad s>2$$

any $f(t)$ that is piecewise continuous has a Laplace transform

↳ finite number of discontinuities on $0<t<\infty$



because $\int_0^\infty f(t)\,e^{-st}\,dt$

$$= \int_0^a \cdots + \int_a^b \cdots + \int_b^c \cdots + \int_c^\infty \cdots$$

for example, $f(t) = \begin{cases} 2t+1 & 0 \le t < 3 \\ e^t & 3 \le t < \infty \end{cases}$

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^3 (2t+1) e^{-st}\, dt + \int_3^\infty e^t e^{-st}\, dt$$

$$\vdots$$

$$= \frac{1}{s} - \frac{7e^{-3s}}{s} + \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} + \frac{e^{-3(s-1)}}{s-1}, \quad s > 1$$

Laplace transform: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

Inverse Laplace transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$= \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

not used in practice

usually we use a table of transforms